

THE CORONA THEOREM AND STABLE RANK FOR THE ALGEBRA $\mathbb{C} + BH^\infty$

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ABSTRACT. Let B be a Blaschke product. We prove in several different ways the corona theorem for the algebra $H_B^\infty := \mathbb{C} + BH^\infty$. That is, we show the equivalence of the classical *corona condition* on data $f_1, \dots, f_n \in H_B^\infty$:

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n |f_k(z)| \geq \delta > 0,$$

and the *solvability of the Bezout equation* for $g_1, \dots, g_n \in H_B^\infty$:

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1.$$

Estimates on solutions to the Bezout equation are also obtained. We also show that the Bass stable rank of H_B^∞ is 1. Let $A(\mathbb{D})_B$ be the subalgebra of all elements from H_B^∞ having a continuous extension to the closed unit disk $\overline{\mathbb{D}}$. Analogous results are obtained also for $A(\mathbb{D})_B$.

1. INTRODUCTION

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. We denote the Hardy algebra of all bounded holomorphic functions on \mathbb{D} (with pointwise operations and the supremum norm) by H^∞ , while the disk algebra is the set of all functions in H^∞ having a continuous extension to $\overline{\mathbb{D}}$ will be denoted by $A(\mathbb{D})$.

It is well known that the unit disk \mathbb{D} is dense in the maximal ideal space of H^∞ . This can be rephrased in terms of function theoretic conditions to say that for any $f_1, \dots, f_n \in H^\infty$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n |f_k(z)| \geq \delta > 0,$$

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there exists $g_1, \dots, g_n \in H^\infty$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1.$$

It is immediate to see that if 1 is in the ideal generated by the functions f_1, \dots, f_n , then the condition

$$\exists \delta > 0 \text{ such that } \forall z \in \mathbb{D}, \quad \sum_{k=1}^n |f_k(z)| \geq \delta > 0$$

is necessary. The fact that this condition is sufficient is the famous Carleson Corona Theorem [4]. In fact, Carleson proved that there are estimates on the solutions g_j in terms of the parameters δ and n , whenever the corona data f_j are in the unit ball of H^∞ .

The goal of this paper is to prove analogous results for certain subalgebras of H^∞ that arise from constraints on the functions, and their derivatives at some points in the disk. With this in mind, we define the space H_B^∞ .

For $a \in \mathbb{D}$, let

$$\varphi_a(z) := \frac{a - z}{1 - \bar{a}z}$$

denote the Möbius transform of the unit disk \mathbb{D} onto itself. Then we define the *Blaschke product* B with zeros a_k of multiplicity m_k by

$$B := \prod_{k \geq 1} \left(\frac{|a_k|}{a_k} \varphi_{a_k} \right)^{m_k} \quad \text{where} \quad \sum_{k \geq 1} m_k (1 - |a_k|) < \infty.$$

Note that the factor $\frac{|a_k|}{a_k}$ is chosen for convergence of the infinite product (if $a_k = 0$ then this factor is -1).

Definition 1.1. Let $a_k, k \geq 1$, denote the zeros in \mathbb{D} of a Blaschke product B with multiplicity m_k . We denote by H_B^∞ the set of all those functions in H^∞ that satisfy

- (A1) For all j, k : $f(a_j) = f(a_k)$;
- (A2) For all k and all $1 \leq m \leq m_k - 1$: $f^{(m)}(a_k) = 0$.

Similarly, $A(\mathbb{D})_B$ is the set of all those functions in $A(\mathbb{D})$ satisfying (A1) and (A2).

We note that $A(\mathbb{D})_B$ reduces to \mathbb{C} if the Blaschke product B has its zeros accumulating at a set of positive Lebesgue measure.

An equivalent and useful definition of the algebra $\mathbb{C} + BH^\infty$ will be the following:

$$H_B^\infty := \mathbb{C} + BH^\infty$$

We will use the definitions interchangeably.

One notes that H_B^∞ and $A(\mathbb{D})_B$ are Banach algebras with the usual point-wise operations and the supremum norm:

$$\|f\|_\infty := \sup_{z \in \mathbb{D}} |f(z)|.$$

Let us point out that for infinite Blaschke products, $\mathbb{C} + BA(\mathbb{D})$ is not an algebra and that $A(\mathbb{D})_B$ is strictly contained in $\mathbb{C} + BA(\mathbb{D})$. For finite Blaschke products both sets coincide.

The algebra H_B^∞ has been studied earlier in [10]. The case when the Blaschke product has a finite number of simple zeros was considered in [11], and the simplest case of one zero at 0 with multiplicity 2 was discussed in [5]. These papers considered Nevanlinna-Pick interpolation problems for these algebras.

These algebras also arise naturally as the restriction of functions in H^∞ and the ball algebra on the bidisc to distinguished varieties. In this context, they have been extensively studied by Agler and McCarthy, see [1] and [2]. Additionally, in the case of a finite Blaschke product, these algebras give examples of finite co-dimensional subalgebras of H^∞ , which were first studied by Gamelin, see [6].

1.1. Main Results. In this article, we prove in four different ways a corona theorem for H_B^∞ . The first version reads as follows:

Theorem 1.2. *The set of multiplicative linear functionals corresponding to evaluations at points of the unit disk is dense in the maximal ideal space of H_B^∞ .*

Note that one proof follows immediately from [7, p.12] and the classical Carleson Corona Theorem, which says that \mathbb{D} is dense in the maximal ideal space of H^∞ [4].)

We also have the following:

Theorem 1.3. *If $A(\mathbb{D})_B \neq \mathbb{C}$, then the maximal ideal space M of $A(\mathbb{D})_B$ is the set of multiplicative linear functionals corresponding to evaluations at points of $\overline{\mathbb{D}}$. M can be identified with the quotient space $\overline{\mathbb{D}}|_\sim$, where $z \sim w$ if and only if $z, w \in \overline{\{a_k : k \in \mathbb{N}\}}$.*

We point out here that because the algebras H_B^∞ and $A(\mathbb{D})_B$ by definition have points where the functions agree, the unit disk does not embed homeomorphically into the spectrum of these algebras. This is because the points where the functions agree give rise to the same linear functionals.

By the standard Gelfand theory of Banach algebras these results imply:

Corollary 1.4. *Let $\mathcal{A} = H_B^\infty$ or $\mathcal{A} = A(\mathbb{D})_B$. Then the following are equivalent:*

- (1) $f_1, \dots, f_n \in \mathcal{A}$ and there exists a $\delta > 0$, such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n |f_k(z)| \geq \delta > 0 \quad (\text{Corona condition}).$$

- (2) There exist $g_1, \dots, g_n \in \mathcal{A}$, such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1 \quad (\text{Bezout equation}).$$

The well known Carleson Corona Theorem for H^∞ is stronger in that it is possible to find estimates for the solution to the Bezout equation in terms of the parameter δ . It is also possible to do this in the case of the algebras H_B^∞ . We first present a “matricial” proof interesting in its own right when B is a *finite* Blaschke product.

Theorem 1.5. *Let B be a finite Blaschke product with zeros a_k of multiplicity m_k , $k = 1, \dots, N$. Let $f_1, \dots, f_n \in H_B^\infty$ be such that*

$$\forall z \in \mathbb{D}, \quad 1 \geq \sum_{k=1}^n |f_k(z)| \geq \delta > 0.$$

Then there exist $g_1, \dots, g_n \in H_B^\infty$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1 \quad \text{and} \quad \forall k \in \{1, \dots, n\}, \quad \|g_k\|_\infty \leq C(n, \delta, a_k, m_k).$$

Using a short argument relying on an application of Carleson’s Corona Theorem, we are able to show that the above theorem holds more generally for algebras generated by ideals in H^∞ . This allows us to remove the dependence upon the points a_k corresponding to the zeros of the Blaschke factors.

Namely, let \mathbb{I} be any ideal in H^∞ , and let

$$H_{\mathbb{I}}^\infty := \{c + \varphi \mid c \in \mathbb{C} \text{ and } \varphi \in \mathbb{I}\}.$$

Then $H_{\mathbb{I}}^\infty$ is a subalgebra of H^∞ . We then establish the following:

Theorem 1.6. *Let \mathbb{I} be a proper ideal in H^∞ and let $f_1, \dots, f_n \in H_{\mathbb{I}}^\infty$ be such that*

$$(1.1) \quad \forall z \in \mathbb{D}, \quad 1 \geq \sum_{k=1}^n |f_k(z)| \geq \delta > 0.$$

Then there exist $g_1, \dots, g_n \in H_{\mathbb{I}}^\infty$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1 \quad \text{and} \quad \forall k \in \{1, \dots, n\}, \quad \|g_k\|_\infty \leq C(n, \delta).$$

We prove Theorem 1.5 and Theorem 1.6 in Section 3 of this article. In Section 4, we show that the Bass stable rank of $A(\mathbb{D})_B$ and H_B^∞ is equal to 1.

2. CORONA THEOREM FOR H_B^∞ AND $A(\mathbb{D})_B$

2.1. The Case of H_B^∞ . Let

$$H_{B0}^\infty := \{f \in H_B^\infty : f(a_1) = 0\}.$$

Note that $H_{B0}^\infty \subset H_B^\infty$, and H_{B0}^∞ is a Banach algebra without identity. It is straightforward to check that H_{B0}^∞ is nothing but the closed ideal BH^∞ generated by the Blaschke product B .

Evaluation at a point $\lambda \in \mathbb{D}$ is a multiplicative linear functional on $H_{B_0}^\infty$ (the trivial one if λ belongs to the zero set Λ of B).

We prove below that in the weak-* topology on the dual space of $H_{B_0}^\infty$, the set of these point evaluations is dense in the set of all multiplicative linear functionals on $H_{B_0}^\infty$.

Lemma 2.1. *The set of multiplicative linear functionals consisting of evaluations at points of $\mathbb{D} \setminus \Lambda$ is dense in the set of all non-zero multiplicative linear functionals on $H_{B_0}^\infty$.*

Proof. Let l be a non-zero multiplicative linear functional on $H_{B_0}^\infty$. Since l is non-zero, there exists a function $p_0 \in H_{B_0}^\infty$ with $l(p_0) \neq 0$. If $f \in H^\infty$, then define

$$L(f) := \frac{l(fp_0)}{l(p_0)}.$$

Note that $(fp_0)(a_k) = f(a_k)0 = 0$ and for $1 \leq m \leq m_k - 1$,

$$(fp_0)^{(m)}(a_k) = f^{(m)}(a_k) \underbrace{p_0(a_k)}_{=0} + \sum_{j=1}^m \binom{m}{j} f^{(m-j)}(a_k) \underbrace{p_0^{(j)}(a_k)}_{=0} = 0.$$

So $fp_0 \in H_{B_0}^\infty$, and L is well-defined. Clearly L is a linear transformation. For $f \in H^\infty$, we have

$$|L(f)| = \frac{|l(fp_0)|}{|l(p_0)|} \leq \frac{\|fp_0\|_\infty}{|l(p_0)|} \leq \left(\frac{\|p_0\|_\infty}{|l(p_0)|} \right) \|f\|_\infty,$$

and so L is continuous on H^∞ . It is also multiplicative, since if $f, g \in H^\infty$, then $l(fgp_0)l(p_0) = l(fgp_0p_0) = l((fp_0)(gp_0)) = l(fp_0)l(gp_0)$, and so dividing by $(l(p_0))^2 (\neq 0)$, we obtain

$$L(fg) = \frac{l(fgp_0)}{l(p_0)} = \frac{l(fp_0)}{l(p_0)} \frac{l(gp_0)}{l(p_0)} = L(f)L(g).$$

Thus L defines a non-zero multiplicative linear functional on H^∞ . By the Carleson Corona Theorem, there exists a net $(\alpha_j)_{j \in J}$ of point evaluations in \mathbb{D} that converges to L in the Gelfand topology of the maximal ideal space of H^∞ . We also observe that l is the restriction of L to $H_{B_0}^\infty$:

$$\forall f \in H_{B_0}^\infty, \quad L(f) = \frac{l(fp_0)}{l(p_0)} = \frac{l(f)l(p_0)}{l(p_0)} = l(f).$$

The restriction of each element in the net $(\alpha_j)_{j \in J}$ to $H_{B_0}^\infty$ gives a net (of point evaluations in \mathbb{D}) that converges to l in the weak-* topology of $H_{B_0}^\infty$. \square

Proof of Theorem 1.2. Let L be a non-zero multiplicative linear functional on H_B^∞ . Let $l := L|_{H_{B_0}^\infty}$. Then l is a multiplicative linear functional on $H_{B_0}^\infty$. If $f \in H_B^\infty$, then $f - f(a_1) \in H_{B_0}^\infty$, and so $L(f) = f(a_1) + l(f - f(a_1))$. Now, if l is identically zero, we have $L(f) = f(a_1)$, and so L is point evaluation (at a_1). On the other hand, if l is non-zero, then by the previous lemma,

there exists a net $(\alpha_j)_{j \in J}$ of point evaluations in \mathbb{D} that converges to l in the weak-* topology of $H_{B_0}^\infty$. Thus for all $f \in H_B^\infty$,

$$\begin{aligned} L(f) &= f(a_1) + l(f - f(a_1)) = f(a_1) + \left(\lim_{j \in J} \alpha_j\right)(f - f(a_1)) \\ &= f(a_1) + \lim_{j \in J} \alpha_j(f - f(a_1)) = f(a_1) + \lim_{j \in J} (f(\alpha_j) - f(a_1)) \\ &= \lim_{j \in J} f(\alpha_j) = \lim_{j \in J} \alpha_j(f). \end{aligned}$$

Thus $L = \lim_{j \in J} \alpha_j$, and this completes the proof. \square

2.2. The Case of $A(\mathbb{D})_B$. The proof for $A(\mathbb{D})_B$ is basically the same, however is technically easier since the maximal ideal space of the disk algebra is $\overline{\mathbb{D}}$, so the language of nets doesn't need to be employed.

Let

$$A(\mathbb{D})_{B_0} := \{f \in A(\mathbb{D})_B : f(a_1) = 0\}.$$

Note that $A(\mathbb{D})_{B_0} \subset A(\mathbb{D})_B$, and $A(\mathbb{D})_{B_0}$ is a Banach algebra without identity. Moreover $A(\mathbb{D})_{B_0} = BH^\infty \cap A(\mathbb{D})$. In order to avoid trivialities, we assume that the zeros of B do not cluster at a set of positive Lebesgue measure, so that $A(\mathbb{D})_{B_0}$ does not collapse to $\{0\}$.

Lemma 2.2. *Each non-zero multiplicative linear functional $l : A(\mathbb{D})_{B_0} \rightarrow \mathbb{C}$ is a point evaluation at some $\lambda \in \overline{\mathbb{D}} \setminus \overline{\Lambda}$.*

Proof. Since l is non-zero, there exists a function $p_0 \in A(\mathbb{D})_{B_0}$ with $l(p_0) \neq 0$. For $f \in A(\mathbb{D})$, define $L(f) = l(fp_0)/l(p_0)$. Then L defines a non-zero multiplicative linear functional on $A(\mathbb{D})$, and hence is point evaluation at some $\lambda \in \overline{\mathbb{D}}$. As l is the restriction of L to $A(\mathbb{D})_{B_0}$, l is point evaluation as well. \square

Proof of Theorem 1.3. Let L be a non-zero multiplicative linear functional on $A(\mathbb{D})_B$. Define $l := L|_{A(\mathbb{D})_{B_0}}$. Then l is a multiplicative linear functional on $A(\mathbb{D})_{B_0}$. If $f \in A(\mathbb{D})_B$, then $L(f) = f(a_1) + l(f - f(a_1))$. Now, if l is identically zero, we have $L(f) = f(a_1)$, and so L is point evaluation (at a_1). On the other hand, if l is non-zero, then by the previous lemma, l is point evaluation, at say, $\lambda \in \overline{\mathbb{D}}$. Thus for all $f \in A(\mathbb{D})_B$, $L(f) = f(a_1) + l(f - f(a_1)) = f(\lambda)$. \square

3. CORONA THEOREM FOR H_B^∞ WITH ESTIMATES

In this section, we show that when the number of zeros of the Blaschke product B is finite, we can get estimates on the size of a solution to the Bezout equation in Corollary 1.4 for H_B^∞ . We also give a different proof that is valid for subalgebras of H^∞ that are generated by ideals, which uses only Carleson's Corona Theorem.

We will use the following simple lemma. Here we use the notation M^\top for the *transpose* of a matrix $M \in \mathbb{C}^{m \times n}$, i.e., $[M^\top]_{ij} = [M]_{ji}$. For a (column)-vector $x \in \mathbb{C}^n$, let $\|x\|$ be the usual Euclidean norm of x ; the norm $\|A\|$ of a matrix $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is given by $\|A\| = (\sum_{i,j} |a_{ij}|^2)^{1/2}$.

Also, in this section we will sometimes use the letter C to denote different constants in the same proof and we will keep track of the parameters that determine the constant.

Lemma 3.1. *Let $x, y \in \mathbb{C}^n$ such that $\|x\| \geq \delta > 0$ and $x^\top y = 0$. Then there exists $A \in \mathbb{C}^{n \times n}$ such that $A = -A^\top$, $Ax = y$ and $\|A\| \leq C(n, \delta)\|y\|$.*

Proof. Since $x \neq 0$, there exists $k \in \{1, \dots, n\}$ such that $x_k \neq 0$ and $|x_k| = \max\{|x_1|, \dots, |x_n|\} \geq C(n)\|x\| \geq C(n)\delta$; for example we may take $C(n) = 1/\sqrt{n}$. Define

$$A := \left[\begin{array}{ccc|c|ccc} 0 & \dots & 0 & \frac{y_1}{x_k} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{y_{k-1}}{x_k} & 0 & \dots & 0 \\ \hline -\frac{y_1}{x_k} & \dots & -\frac{y_{k-1}}{x_k} & 0 & -\frac{y_{k+1}}{x_k} & \dots & -\frac{y_n}{x_k} \\ \hline 0 & \dots & 0 & \frac{y_{k+1}}{x_k} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{y_n}{x_k} & 0 & \dots & 0 \end{array} \right].$$

Then $A = -A^\top$, and using $y_1 x_1 + \dots + y_n x_n = 0$, it can be verified that $Ax = y$. Finally, we note that $\|A\| \leq C\|y\|/|x_k| \leq C(n, \delta)\|y\|$. \square

Let us note that whenever M is a matrix in $\mathbb{C}^{n \times n}$ such that $M = -M^\top$, then $(Mx)^\top x = 0$ for every $x \in \mathbb{C}^n$. We will also need the following result on polynomial interpolation:

Lemma 3.2. *Given $a_k \in \mathbb{D}$, and $m_k \geq 1$, $\alpha_k^{(m)} \in \mathbb{C}$, $0 \leq m \leq m_k - 1$, $k = 1, \dots, N$, there exists a polynomial p such that*

$$(3.1) \quad p^{(m)}(a_k) = \alpha_k^{(m)}, \quad k = 1, \dots, N, \quad 0 \leq m \leq m_k - 1.$$

Proof. For $k \in \{1, \dots, N\}$, let

$$\begin{aligned} p_k(z) &:= (z - a_1)^{m_1} \dots (z - a_{k-1})^{m_{k-1}} (z - a_{k+1})^{m_{k+1}} \dots (z - a_N)^{m_N}, \\ q_k(z) &:= 1/p_k(z). \end{aligned}$$

Define for $1 \leq k \leq N$, $0 \leq m \leq m_k - 1$:

$$P_{k,m}(z) := p_k(z) \frac{(z - a_k)^m}{m!} \sum_{l=0}^{m_k-1-m} \frac{q_k^{(l)}(a_k)}{l!} (z - a_k)^l,$$

Let

$$p(z) := \sum_{k=1}^N \sum_{m=0}^{m_k-1} \alpha_k^{(m)} P_{k,m}(z).$$

That p satisfies (3.1) can be verified in the same manner as the proof of Theorem 1 in [12]. \square

With these two lemmas, it is now possible to give the proof of Theorem 1.5.

Recall that for a vector $f = (f_1, \dots, f_n) \in (H^\infty)^n$, we define its norm by $\|f\|_\infty := \max_{1 \leq j \leq n} \|f_j\|_\infty$. Similarly for a matrix $M = (f_{ij})_{1 \leq i, j \leq n} \in (H^\infty)^{n \times n}$.

Proof of Theorem 1.5. By Carleson's Corona theorem, there exists a $g_0 \in (H^\infty)^n$ such that

$$g_0 \cdot f = 1,$$

and $\|g_0\|_\infty \leq C(\delta)$. We will find a suitable matrix $H \in (H^\infty)^{n \times n}$ such that

- (1) $H = -H^\top$,
- (2) (with norm control) $\|H\|_\infty \leq C(n, \delta, a_k, m_k)$, and
- (3) $g := g_0 + Hf \in (H_B^\infty)^n$.

Then clearly $g \cdot f = 1$, giving Theorem 1.5.

Let $k \geq 1$. Since $g_0 \cdot f = 1$ and $f(a_k) = f(a_1)$, we have

$$g_0(a_1) \cdot f(a_1) = 1 \quad \text{and} \quad g_0(a_k) \cdot f(a_1) = 1.$$

Thus

$$(g_0(a_k) - g_0(a_1)) \cdot f(a_1) = 0.$$

Note that with $g := g_0 + Hf$, we have

$$g(a_1) = g(a_k) \quad \text{iff} \quad (H(a_k) - H(a_1))f(a_1) = g_0(a_1) - g_0(a_k).$$

Choose $A_{a_1}^{(0)} \in \mathbb{C}^{n \times n}$ such that

$$(A_{a_1}^{(0)})^\top = -A_{a_1}^{(0)}.$$

Next choose $A_{a_k}^{(0)} \in \mathbb{C}^{n \times n}$ such that

$$(A_{a_k}^{(0)})^\top = -A_{a_k}^{(0)} \quad \text{and} \quad (A_{a_k}^{(0)} - A_{a_1}^{(0)})f(a_1) = g_0(a_1) - g_0(a_k),$$

which is possible in light of Lemma 3.1 and the choice of $A_{a_1}^{(0)}$. We also note that since $\|g_0\|_\infty \leq C(\delta)$, we can estimate $\|A_{a_k}^{(0)}\| \leq C(n, \delta)$.

Since $g_0 \cdot f = 1$, by differentiating this m times, with $1 \leq m \leq m_k - 1$, and evaluating at a_k , we obtain

$$\sum_{j=0}^m \binom{m}{j} g_0^{(m-j)}(a_k) \cdot f^{(j)}(a_k) = 0.$$

Thus

$$g_0^{(m)}(a_k) \cdot f(a_k) = 0.$$

We note that with $g := g_0 + Hf$, we have

$$g^{(m)}(a_k) = 0 \quad \text{if and only if} \quad H^{(m)}(a_k)f(a_k) = -g_0^{(m)}(a_k).$$

Again, using Lemma 3.1, we can choose $A_{a_k}^{(m)} \in \mathbb{C}^{n \times n}$ such that

$$(A_{a_k}^{(m)})^\top = -A_{a_k}^{(m)} \quad \text{and} \quad A_{a_k}^{(m)}f(a_k) = -g_0^{(m)}(a_k).$$

We note that by Cauchy integral formula, $|g_0^{(m)}(a_k)| \leq C(a_k, m_k) \|g_0\|_\infty$, and so by Lemma 3.1, $\|A_{a_k}^{(m)}\| \leq C(n, \delta, a_k, m_k)$.

We can find $\frac{n(n-1)}{2}$ functions $h_{ij} \in H^\infty$, $2 \leq i \leq j \leq n$ such that

$$h_{ij}^{(m)}(a_k) = [A_{a_k}^{(m)}]_{ij}, \quad 0 \leq m \leq m_k - 1, \quad k = 1, \dots, n$$

and

$$\|h_{ij}\|_\infty \leq C(\sup\{|[A_{a_k}^{(m)}]_{ij}| : 0 \leq m \leq m_k - 1, \quad k = 1, \dots, n\}).$$

Indeed, interpolating polynomials as in Lemma 3.2 above will do. Define

$$H := \begin{bmatrix} 0 & h_{1,2} & h_{1,3} & \dots & h_{1,n} \\ -h_{1,2} & 0 & h_{2,3} & \dots & h_{2,n} \\ -h_{1,3} & -h_{2,3} & 0 & \dots & h_{3,n} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ -h_{1,n-1} & \dots & \dots & 0 & h_{n-1,n} \\ -h_{1,n} & \dots & \dots & -h_{n-1,n} & 0 \end{bmatrix} \in (H^\infty)^{n \times n}.$$

Then $g := g_0 + Hf \in H_B^\infty$, $g \cdot f = 1$ and $\|g\|_\infty \leq C(n, \delta, a_k, m_k)$. \square

3.1. Generalisation to arbitrary ideals. Recall that $H_{\mathbb{I}}^\infty$ is a subalgebra of H^∞ given by \mathbb{I} , an ideal in H^∞ , with

$$H_{\mathbb{I}}^\infty := \{c + \varphi \mid c \in \mathbb{C} \text{ and } \varphi \in \mathbb{I}\}.$$

We now will show that the Corona Theorem holds for these algebras.

Proof of Theorem 1.6. Let $f_k = c_k + \varphi_k$, where $c_k \in \mathbb{C}$ and $\varphi_k \in \mathbb{I}$ for all $k \in \{1, \dots, n\}$. Since $\mathbb{I} \neq H^\infty$, we see that the ideal $(\varphi_1, \dots, \varphi_n)$ generated by the φ_j 's is proper; hence $\inf_{z \in \mathbb{D}} \sum_{j=1}^n |\varphi_j(z)| = 0$. Thus there exists a sequence $(z_k)_{k \in \mathbb{N}}$ in \mathbb{D} so that $\varphi_j(z_k) \rightarrow 0$ for every j . Hence

$$\delta \leq \sum_{j=1}^n |f_j(z_k)| = \sum_{j=1}^n |c_j + \varphi_j(z_k)| \rightarrow \sum_{j=1}^n |c_j|.$$

Therefore (1.1) gives

$$1 \geq |c_1| + \dots + |c_n| \geq \delta.$$

So all c_k 's satisfy $|c_k| \leq 1$ and at least one of the c_k 's (which we may assume is $k = 1$ without loss of generality) satisfies $|c_k| \geq \delta/n$. By Carleson's Corona Theorem, there exist $\tilde{g}_1, \dots, \tilde{g}_n \in H^\infty$ such that

$$1 = f_1 \tilde{g}_1 + \dots + f_n \tilde{g}_n \text{ on } \mathbb{D},$$

and $\|\tilde{g}_k\|_\infty \leq C(\delta)$ for all k 's. Thus,

$$\begin{aligned} 1 &= \frac{1}{c_1}(f_1 - \varphi_1) = \frac{1}{c_1}(f_1 - 1\varphi_1) = \frac{1}{c_1}\left(f_1 - \left(\sum_{k=1}^n f_k \tilde{g}_k\right)\varphi_1\right) \\ &= f_1\left(\frac{1}{c_1} - \tilde{g}_1\varphi_1\right) + f_2\left(-\frac{1}{c_1}\tilde{g}_2\varphi_1\right) + \cdots + f_n\left(-\frac{1}{c_1}\tilde{g}_n\varphi_1\right) \\ &= f_1g_1 + f_2g_2 + \cdots + f_ng_n, \end{aligned}$$

where

$$g_1 := \frac{1}{c_1} - \tilde{g}_1\varphi_1, \quad g_2 := -\frac{1}{c_1}\tilde{g}_2\varphi_1, \quad \dots, \quad g_n := -\frac{1}{c_1}\tilde{g}_n\varphi_1.$$

Clearly $g_1, \dots, g_n \in H_{\mathbb{I}}^\infty$, and since we have $\frac{1}{c_1} \leq \frac{n}{\delta}$ and

$$\|\varphi_1\|_\infty = \|f_1 - c_1\|_\infty \leq \|f_1\|_\infty + |c_1| \leq 1 + 1 = 2,$$

it follows that $\|g_k\|_\infty \leq C(n, \delta)$ for all k . \square

4. STABLE RANKS

The notion of stable rank of a ring (which we call Bass stable rank) was introduced by H. Bass [3] to facilitate computations in algebraic K-theory. We recall the definition of the Bass stable rank of a ring below.

Definition 4.1. Let A be an algebra with a unit e . An n -tuple $a \in A^n$ is called *unimodular* if there exists an n -tuple $b \in A^n$ such that $\sum_{j=1}^n a_j b_j = e$. An n -tuple a is called *stable* or *reducible* if there exists an $(n-1)$ -tuple x such that the $(n-1)$ -tuple $(a_1 + x_1 a_n, \dots, a_{n-1} + x_{n-1} a_n)$ is unimodular. The *stable rank* (also called *bsr*(A) in the literature) of the algebra A is the least integer n such that every unimodular $(n+1)$ -tuple is reducible.

It is straightforward to see that for the algebras $A(\mathbb{D})$ and $H^\infty(\mathbb{D})$ that unimodular tuples are the same as Corona data in the algebra.

Over the last two decades there has been some interest in studying the Bass stable rank of Banach algebras. Jones, Marshall and Wolff [9] showed that the Bass stable rank of the disk algebra $A(\mathbb{D})$ is one, and Treil [14] proved that the Bass stable rank of H^∞ is one as well; see also [13]:

Proposition 4.2 (Jones-Marshall-Wolff). *The Bass stable rank of $A(\mathbb{D})$ is 1.*

Proposition 4.3 (Treil). *The Bass stable rank of H^∞ is 1.*

Using Treil's result, an analogous result holds for the subalgebra $H_{\mathbb{I}}^\infty = \mathbb{C} + \mathbb{I}$:

Theorem 4.4. *Let \mathbb{I} be a proper ideal in H^∞ . Then the Bass stable rank of $H_{\mathbb{I}}^\infty$ is 1.*

Proof. Let $(f, g) = (a + \varphi, b + \psi)$ be a unimodular pair in $H_{\mathbb{I}}^{\infty}$, where $a, b \in \mathbb{C}$ and $\varphi, \psi \in \mathbb{I}$. In particular $\delta := \inf_{z \in \mathbb{D}} |f(z)| + |g(z)| > 0$. We consider the two cases $a \neq 0$ and $a = 0$.

1° If $a \neq 0$, then the pair, $(f, g\varphi)$ is unimodular. Indeed, if $z \in \mathbb{D}$ is such that $|\varphi(z)| \geq |a|/2$, then $|f| + |g\varphi| \geq \min\{1, |a|/2\}(|f| + |g|) \geq \delta \min\{1, |a|/2\}$ at that z , while if $|\varphi(z)| \leq |a|/2$, then $|f| = |a + \varphi| \geq |a| - |a|/2 = |a|/2$, so that $|f| + |g\varphi| \geq |a|/2$. By Treil's Theorem, the element $f + h(g\varphi)$ is invertible in H^{∞} for some element $h \in H^{\infty}$. But writing $f + h(g\varphi) = f + (h\varphi)g$, and noting that $h\varphi \in H_{\mathbb{I}}^{\infty}$, we are done since $f + (h\varphi)g$ being invertible in H^{∞} , is also invertible in $H_{\mathbb{I}}^{\infty}$ by the corona theorem for $H_{\mathbb{I}}^{\infty}$.

2° If $a = 0$, then clearly $b \neq 0$, as (f, g) satisfies the corona condition. We know that there exist $x, y \in H_{\mathbb{I}}^{\infty}$ such that $fx + gy = 1$. Thus $(f + g)x + g(y - x) = 1$, and so $(f + g, g)$ is unimodular. By case 1° above, it follows that there exists a $h \in H_{\mathbb{I}}^{\infty}$ such that $(f + g) + hg$ is invertible in $H_{\mathbb{I}}^{\infty}$, that is, $f + (1 + h)g$ is invertible in $H_{\mathbb{I}}^{\infty}$. Since $1 + h \in H_{\mathbb{I}}^{\infty}$, we are done. \square

By using the result of Jones, Marshall and Wolff that the Bass stable rank of the disk algebra $A(\mathbb{D})$ is one, we now establish the following:

Theorem 4.5. *Let B be a Blaschke product. Suppose that the zeros of B do not cluster at a set of positive Lebesgue measure (hence $A(\mathbb{D})_B$ is not trivial). Then the Bass stable rank of $A(\mathbb{D})_B$ is 1.*

Proof. Recall that $A(\mathbb{D})_B = (\mathbb{C} + BH^{\infty}) \cap A(\mathbb{D})$. Let $(f, g) = (a + BF, b + BG)$ be a unimodular pair in $A(\mathbb{D})_B$. Note that F and G necessarily are in $A(\mathbb{D})$. As above, we consider two cases:

1° If $a \neq 0$, then we let E be the set of cluster points of the zero set of B (if B is a finite Blaschke product, then $E = \emptyset$). Since E has Lebesgue measure zero, there exists by [8, p. 81] a function $p \in A(\mathbb{D})$ that satisfies $p = 1$ on E and $|p| < 1$ on $\overline{\mathbb{D}} \setminus E$. (If $E = \emptyset$, we take $p \equiv 0$.) Then $B(1 - p) \in A(\mathbb{D})_B$. Consider the pair $(f, B(1 - p)g)$. Then this is unimodular, too. Since $A(\mathbb{D})$ has the stable rank 1, there exists $h \in A(\mathbb{D})$ such that $f + hB(1 - p)g$ is invertible in $A(\mathbb{D})$. But $B(1 - p)h \in A(\mathbb{D})_B$. So $f + hB(1 - p)g \in A(\mathbb{D})_B$. Thus the pair (f, g) is reducible in $A(\mathbb{D})_B$.

2° If $a = 0$, then we consider the unimodular pair $(f + g, g)$ and use case 1° to conclude that $(f + g, g)$ and hence (f, g) is reducible in $A(\mathbb{D})_B$. \square

We remark at this point that it is possible to prove a general algebraic proposition along the same lines. Namely, let \mathcal{A} be a commutative unital, normed algebra and let \mathbb{I} be a non-zero ideal in \mathcal{A} . If \mathcal{A} has Bass stable rank n , then $\mathbb{C} + \mathbb{I}$ has Bass stable rank n as well.

5. CONCLUDING REMARKS

Analysing the proof Theorem 1.6, it seems reasonable to make the following conjecture, where the new feature is that the constant $C(\delta)$ is independent of the number n of generators. This conjecture is inspired by the fact that the corresponding estimates in H^∞ are true.

Conjecture 5.1. *Let B be a Blaschke product with infinitely many zeros a_k of multiplicity m_k . Suppose that $f_1, \dots, f_n \in H_B^\infty$ satisfy*

$$\forall z \in \mathbb{D}, \quad 1 \geq \sum_{k=1}^n |f_k(z)| \geq \delta > 0.$$

Then there exist $g_1, \dots, g_n \in H_B^\infty$ such that

$$\forall z \in \mathbb{D}, \quad \sum_{k=1}^n g_k(z) f_k(z) = 1 \quad \text{and} \quad \forall k \in \{1, \dots, n\}, \quad \|g_k\|_\infty \leq C(\delta).$$

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